International Journal of Theoretical Physics, Vol. 40, No. 5, 2001

Liquid Helium-4 in the Static Fluctuation Approximation

M. K. Al-Sugheir,[1,4] H. B. Ghassib,[2] and R. R. Nigmatullin[3]

Received October 10, 2000

In this work liquid helium-4 is studied for the first time within the framework of the so-called *static fluctuation approximation.* This is based on the replacement of the *square* of the local-field operator with its mean value. A closed set of nonlinear integral equations is derived for weakly as well as for strongly interacting systems. This set is solved numerically by an iteration method for a realistic interhelium potential. The thermodynamic properties are then obtained for both the weakly interacting system, liquid ^{4}He in Vycor glass, and the strongly interacting system, liquid ^{4}He. It turns out, however, that the present quadratic-fluctuation approximation is valid in the latter, strongly interacting case only in the low-temperature limit ($\leq$0.15 K). Our results are presented in a set of figures. The role of the interaction is emphasized and the functional dependence of key thermodynamic quantities on the temperature is derived for both weakly and strongly interacting ^{4}He systems.

1. INTRODUCTION

The extraordinary properties of superfluid ^{4}He (He II) have challenged physicists for more than two-thirds of a century (Galasiewicz, 1971; Ghassib and Khudeir, 1986; Girish and Yia-Chung, 1998; Tilley and Tilley, 1974; Wilks, 1967). Many theoretical techniques and formalisms have been applied to explain the observed behaviour of the system, but a comprehensive microscopic theory is still lacking.

In this paper we shall use a new approach, applied here for the first time to many-bosonic systems, namely, the so-called *static fluctuation approximation* (SFA) (Nigmatullin and Toboev, 1989). This approach has recently been used to study the classical two-dimensional Ising model (Nigmatullin *et al.,* 2000a); the

---

[1] Department of Physics, The Hashemite University, Zarqa, Jordan.
[2] Department of Physics, University of Jordan, Amman, Jordan.
[3] Theoretical Physics Department, Kazan State University, Kazan, 420008, Tatarstan, Russia.
[4] To whom correspondence should be addressed at Department of Physics, Faculty of Science and Arts, The Hashemite University, Zarqa 13115, Jordan; e-mail: hghassib@nic.net.jo

one-, two-, and three-dimensional Ising model (Nigmatullin *et al.*, 2000b); and the proton model of ferroelectrics with tunneling (Nigmatullin *et al.*, 2000c). The SFA attempts to rectify the difficulties plaguing conventional many-body theories, exploring at the same time the extreme simplicity of the modified mean-field approximation. The approach is relatively simple compared to other many-body approaches: it is not based on Green's functions or Feynman diagrams, and it is applicable—at least in principle—to any arbitrary system (weakly or strongly interacting).

The underlying key idea is to replace the square of the local-field operator with its mean value. The physical implication is that the true quantum-mechanical spectrum of this operator is replaced with a distribution around the expectation value of the local-field operator (Nigmatullin *et al.*, 2000a,b,c).

It will be seen that such an approximation is sufficient for calculating the equilibrium correlation functions and the main thermodynamic characteristics of the system considered. To this end, it is necessary to obtain the self-consistent so-called *long-range equation* for a wide range of interactions.

In this paper, we shall consider an extended system of N ^{4}He atoms, each of mass m, occupying a volume Ω. In a uniform infinite system, all physical properties must be invariant under spatial translation. This translational invariance implies that the single-particle wavefunctions are plane waves. The constituent bosons are the elementary particles of the system, the energy required for an internal excitation being much larger than the thermal energy.

The rest of the paper is arranged as follows. Section 2 summarizes the basic principles and techniques of SFA. This is followed, in Section 3, by a full derivation of the closed set of nonlinear integral equations involved. Next (Section 4) come the calculations and numerics, then (Section 5) the results for both the weakly interacting system (^{4}He in Vycor) and the strongly interacting system (liquid ^{4}He). Finally, some concluding remarks close the paper (Section 6).

2. BASIC PRINCIPLES AND TECHNIQUES OF SFA

For a specific Hamiltonian $\hat{H}$, the Heisenberg representation of a creation operator $\hat{b}_k^+(\tau)$ is given by

$$\hat{b}_k^+(\tau) = \exp(\tau\hat{H})\hat{b}_k^+(0)\exp(-\tau\hat{H}), \tag{1}$$

where $\tau \equiv it$. The equation of motion of the creation operator in this representation can be written in the form

$$\frac{d\hat{b}_k^+(\tau)}{d\tau} = \left[\hat{H}, \hat{b}_k^+(\tau)\right]. \tag{2}$$

Here k is an index denoting the complete set of compatible quantum numbers describing a specific state. We shall assume that the Hamiltonian can be expressed as

$$\hat{H} = \sum_{\vec{k}} \hat{w}_k \hat{b}_k^+ \hat{b}_k, \tag{3}$$

$\hat{w}_k(\tau)$ being the local-field operator assumed to be hermitian; it commutes with creation and annihilation operators according to the well-known relations for a Bose system:

$$\lfloor \hat{b}_k, \hat{b}_q^+ \rfloor = \delta_{kq} \quad \text{and} \quad \lfloor \hat{b}_k^+, \hat{b}_q^+ \rfloor = 0.$$

We can rewrite (2) as

$$\frac{d\hat{b}_k^+(\tau)}{d\tau} = \left[\hat{H}, \hat{b}_k^+(\tau)\right] = \hat{w}_k(\tau)\, \hat{b}_k^+(\tau). \tag{4}$$

In the well-known mean-field approximation, the local-field operator $\hat{w}_k(\tau)$ is replaced with its mean value:

$$\hat{w}_k(\tau) \cong \langle \hat{w}_k(\tau) \rangle = \langle \hat{w}_k(0) \rangle. \tag{5}$$

In the SFA, we shall go further and assume that the *square* of the local-field operator $\hat{w}_k(\tau)$ can be replaced with its mean value:

$$\hat{w}_k^2(\tau) \cong \left\langle \hat{w}_k^2(\tau) \right\rangle = \left\langle \hat{w}_k^2(0) \right\rangle = \Omega_k^2. \tag{6}$$

This is the single key assumption in this approximation. The implication is that not only the mean value of the spectrum is taken into account (the mean-field approximation), but also the fluctuations of the spectrum around the mean value of the local-field operator. In other words, the true quantum-mechanical spectrum of the local-field operator is replaced with a distribution around the expectation value of the local-field operator (Nigmatullin and Toboev, 1989; Nigmatullin *et al.*, 2000a,b,c).

From approximation (6) it follows that the local-field operator $\hat{w}_k(\tau)$ is time-independent:

$$w_k(\tau) \frac{dw_k(\tau)}{d\tau} = 0. \tag{7}$$

Therefore $\hat{w}_k(\tau)$ is a constant of motion; this is the origin of the qualifier “static” in the SFA:

$$[\hat{H}, \hat{w}_k] = 0. \tag{8}$$

From (4) we obtain

$$\hat{b}_k^+(\tau) = \hat{b}_k^+(0) \exp(\hat{w}_k \tau). \tag{9}$$

From this solution one can obtain all the necessary equilibrium correlation functions. For this purpose it is sufficient to write the identity

$$\langle \hat{A}(\beta)\hat{B}\rangle = \langle \hat{B}\hat{A}\rangle = \frac{1}{Q} Tr\,(\exp(-\beta\hat{H})\hat{B}\hat{A}), \tag{10}$$

where Q is the grand partition function of the system. Here and below we shall replace τ with $\beta \equiv \frac{1}{k_{\mathrm{B}}T}$, k_{B} being Boltzmann's constant and T the absolute temperature; this is common in statistical quantum mechanics (Feynman, 1972).

Using the above commutation rules, we obtain the long-range equation for this case, which relates the mean value of the operator $\hat{n}_k \equiv \hat{b}_k^+\hat{b}_k$ to the mean value of $\hat{w}_k$. According to (10),

$$\left\langle \hat{b}_k^+(\beta)\hat{b}_k\hat{A}\right\rangle = \left\langle \hat{b}_k\hat{A}\hat{b}_k^+\right\rangle. \tag{11}$$

If the operator $\hat{A}$ is chosen to commute with $\hat{b}_k^+$ and $\hat{b}_k$, (11) can then be written as:

$$\left\langle \hat{b}_k^+(\beta)\hat{b}_k\hat{A}\right\rangle = \left\langle \hat{b}_k\hat{A}\hat{b}_k^+\right\rangle = \left\langle \hat{b}_k\hat{b}_k^+\hat{A}\right\rangle. \tag{12}$$

From (9) and (12) we can obtain the long-range equation as

$$\langle \hat{n}_k(\exp(\beta\hat{w}_k) - 1)\hat{A}\rangle = \langle \hat{A}\rangle. \tag{13}$$

This yields the generalized Hartree equations by adding the scalar energy ε_k to the operator $\hat{w}_k$. The Heisenberg equation of motion for $\hat{b}_k^+(\tau)$ becomes

$$\frac{d\hat{b}_k^+(\tau)}{d\tau} = \left[\hat{H}, \hat{b}_k^+(\tau)\right] = (\varepsilon_k + \hat{w}_k(\tau))\hat{b}_k^+(\tau). \tag{14}$$

The corresponding general solution is

$$\hat{b}_k^+(\tau) = \hat{b}_k^+(0)\exp((\varepsilon_k + \hat{w}_k)\tau). \tag{15}$$

Repeating all the above manipulations, we finally obtain

$$\langle \hat{n}_k(\exp(\beta[\varepsilon_k + \hat{w}_k]) - 1)\hat{A}\rangle = \langle \hat{A}\rangle. \tag{16}$$

3. CLOSED SET OF NONLINEAR INTEGRAL EQUATIONS FOR A NEUTRAL BOSE SYSTEM

The total Hamiltonian describing the neutral many-bosonic system can be written as the sum of two terms:

$$\hat{H} = \hat{H}_0 + \hat{H}_1, \tag{17}$$

where $\hat{H}_0$ is the kinetic energy term,

$$\hat{H}_0 = \int d\vec{r}\,\hat{\Psi}^+(\vec{r})\left(-\frac{\hbar^2}{2m}\nabla^2\right)\hat{\Psi}(\vec{r}); \tag{18}$$

$\hat{\Psi}(\vec{r})$ and $\hat{\Psi}^{+}(\vec{r})$ are the field operators; h is Planck's constant ($\hbar \equiv h/2\pi \equiv$ Dirac's constant); m is the bosonic mass; and $\hat{H}_1$ is the interaction term:

$$\hat{H}_1 = \frac{1}{2}\iint d\vec{r}_1 d\vec{r}_2 \hat{\Psi}^{+}(\vec{r}_1)\hat{\Psi}^{+}(\vec{r}_2)V(\vec{r}_1 - \vec{r}_2)\hat{\Psi}(\vec{r}_2)\hat{\Psi}(\vec{r}_1), \tag{19}$$

$V(\vec{r}_1 - \vec{r}_2)$ being the pairwise central potential that depends only on the modulus of $|\vec{r}_1 - \vec{r}_2|$.

It is convenient to write the field operators as linear combinations of the creation and annihilation operators:

$$\hat{\Psi}(\vec{r}) = \sum_{\vec{k}} \Psi_{\vec{k}}(\vec{r})\hat{b}_k; \tag{20}$$

$$\hat{\Psi}^{+}(\vec{r}) = \sum_{\vec{k}} \Psi^{+}_{\vec{k}}(\vec{r})\hat{b}^{+}_k, \tag{21}$$

where the coefficients $\Psi_{\vec{k}}(\vec{r})$, $\Psi^{+}_{\vec{k}}(\vec{r})$ are the single-particle wavefunctions and the sum is over the complete set of single-particle quantum numbers. In particular, the index $\vec{k}$ for spinless bosons merely denotes the linear momentum of the particle.

In a uniform homogeneous system, all physical properties must, of course, be invariant under spatial translations. This suggests periodic boundary conditions and single-particle wavefunctions that are plane waves.

We shall now rewrite (17) in the second-quantization representation:

$$\hat{H} = \sum_{\vec{k}} \frac{\hbar^2 k^2}{2m}\hat{b}^{+}_k \hat{b}_k + \frac{1}{2}\sum_{\vec{k}} V(k)\hat{\rho}_k\hat{\rho}_{-k}, \tag{22}$$

where $V(k)$ is the Fourier transform of the pair potential defined as

$$V(k) = \int V(r)\exp(-\vec{k}\cdot\vec{r})\,d\vec{r}; \qquad \hat{\rho}_k \equiv \frac{1}{\sqrt{\Omega}}\sum_{\vec{q}} \hat{b}^{+}_{k+q}\hat{b}_q, \tag{23}$$

Ω being the normalization volume of the system.

The single-particle energy for free bosons, incorporating the chemical potential, is

$$\varepsilon(k) = \frac{\hbar^2 k^2}{2m} - \mu, \tag{24}$$

μ being the chemical potential: it is the energy required to remove (add) a particle from (to) the system at constant volume. This is equivalent to the particle being in a potential well with depth $-\mu$. Thus, the above Hamiltonian should be modified to incorporate μ:

$$\hat{H} = \sum_{\vec{k}} \varepsilon(k)\hat{b}^{+}_k \hat{b}_k + \frac{1}{2}\sum_{\vec{k}} V(k)\hat{\rho}_k\hat{\rho}_{-k}. \tag{25}$$

We shall assume that we can write this Hamiltonian as a linear combination of the local-field operator $\hat{E}_k$ and the number-of-particles operator:

$$\hat{H} = \sum_{\vec{k}} \hat{E}_k \hat{b}_k^+ \hat{b}_k, \tag{26}$$

where $\hat{E}_k$ is to be found from the equation of motion for the operator $\hat{b}_k^+$ in the Heisenberg picture, and from (25),

$$\frac{d\hat{b}_k^+}{d\tau} = \left[\hat{H}, \hat{b}_k^+\right] = \hat{E}_k \hat{b}_k^+. \tag{27}$$

First,

$$\left[\hat{H}, \hat{b}_k^+\right] = \hat{E}_k \hat{b}_k^+ = \varepsilon(k) b_k^+ + \frac{1}{2\sqrt{\Omega}} \sum_{\vec{q}} V(q) \left(\hat{\rho}_q \hat{b}_{k-q}^+ + \hat{b}_{q+k}^+ \hat{\rho}_{-q}\right). \tag{28}$$

Second,

$$\hat{E}_k = \left[\hat{b}_k, \left[\hat{H}, \hat{b}_k^+\right]\right] = \varepsilon(k) + \frac{1}{\Omega} \sum_{\vec{q}} W(\vec{k}, \vec{q}) \hat{n}_q; \tag{29}$$

$$W(\vec{k}, \vec{q}) \equiv V(0) + V(\vec{k} - \vec{q}). \tag{30}$$

The well-known mean-field approximation states that the local-field operator can be replaced with its mean value; it assumes that the fluctuations in this operator are negligible. In the SFA the square of the quadratic fluctuation operator can be replaced with its mean value:

$$(\Delta \hat{E}_k)^2 \cong \langle (\Delta \hat{E}_k)^2 \rangle, \tag{31}$$

where $\Delta \hat{E}_k \equiv \hat{E}_k - \langle \hat{E}_k \rangle$ is the corresponding deviation of the mean-field operator from its mean value. It follows that the local-field deviation operator has two symmetric eigenvalues:

$$(\Delta \hat{E}_k)^2 \cong \langle (\Delta \hat{E}_k)^2 \rangle = {\varphi_k}^2, \tag{32}$$

which, in fact, defines a splitting of the operator $\Delta \hat{E}_k$ produced by the quadratic fluctuations.

In the SFA the local-field operator $\hat{E}_k$ must commute with both the creation ($\hat{b}_k^+$) and annihilation ($\hat{b}_k$) operators if our closed system of nonlinear integral equations is to be obtained. From (29) we have found that this operator commutes with the creation and annihilation operators if the potential in momentum space vanishes when the relative momentum vanishes, that is, $[V(k = 0) = 0]$. This condition is not valid in arbitrary neutral many-bosonic systems, such as liquid ^{4}He, where the potential has a maximum when the relative momentum vanishes and decreases when the relative momentum increases. Therefore, we shall rewrite the

Hamiltonian so as to be consistent with the condition that $\hat{E}_k$ commutes with $\hat{b}_k^+$ and $\hat{b}_k$:

$$\hat{H} = \sum_{\vec{q}} \varepsilon(q)\hat{n}_q + \frac{1}{2}\sum_{\vec{q}\neq 0} V(q)\hat{\rho}_q\hat{\rho}_{-q} + \frac{\hat{N}^2}{2\Omega}V(0); \tag{33a}$$

or

$$\hat{H} = \sum_{\vec{q}} \varepsilon(q)\hat{n}_q + \frac{1}{2}\sum_{\vec{q}\neq 0} V(q)\hat{\rho}_q\hat{\rho}_{-q} + \frac{N^2}{2\Omega}V(0). \tag{33b}$$

The square of the total number-of-particles operator in (33a) has been replaced with its mean value, where the total number of particles is very large. In this case $\hat{E}_k$ is

$$\hat{E}_k = \left[b_k, \left[\hat{H}, b_k^+\right]\right] = \varepsilon(k) + \frac{1}{\Omega}\sum_{\vec{q}\neq\vec{k}} V(\vec{k}-\vec{q})\hat{n}_q; \tag{34}$$

$$\Delta\hat{E}_k = \frac{1}{\Omega}\sum_{q\neq k} V(\vec{k}-\vec{q})\Delta\hat{n}_q. \tag{35}$$

Operator $\hat{E}_k$ satisfies the following commutation rules:

$$[\hat{E}_k, \hat{b}_k] = 0; \qquad \left[\hat{E}_k, \hat{b}_k^+\right] = 0.$$

Using (16) and letting $\varepsilon_k \to \langle\hat{E}_k\rangle$, $\hat{w}_k \to \Delta\hat{E}_k$, we obtain the long-range equation

$$\langle\hat{n}_k(\exp\beta(\langle\hat{E}_k\rangle + \Delta\hat{E}_k) - 1)\hat{A}\rangle = \langle\hat{A}\rangle. \tag{36}$$

If the operator $\hat{A}$ is chosen to commute with $\hat{E}_k$, and if the fluctuations $\Delta\hat{E}_k$ are small compared to the mean value of the local-field operator $\langle\hat{E}_k\rangle$, then (36) can be rewritten in the form

$$\langle\hat{n}_k\hat{A}\rangle = \left\langle\frac{\hat{A}}{\exp\beta(\langle\hat{E}_k\rangle + \Delta\hat{E}_k) - 1}\right\rangle. \tag{37}$$

It is desirable to write the long-range equation (37) as linear in terms of the fluctuations of the local-field operator; this could be done with the aid of the identity

$$B(a + b\Delta\hat{E}_k) \equiv \eta_0(k) + \eta_1(k)\Delta\hat{E}_k, \tag{38}$$

with

$$\eta_0(k) \equiv \frac{1}{2}\left(B(a + b\varphi_k) + B(a - b\varphi_k)\right); \tag{39a}$$

$$\eta_1(k) \equiv \frac{1}{2\varphi_k}(B(a + b\varphi_k) - B(a - b\varphi_k)), \tag{39b}$$

which is correct for any smooth function $B(x)$. According to this identity, one can write (37) in the form

$$\langle \hat{n}_k \hat{A} \rangle = \eta_0(k)\langle \hat{A} \rangle + \eta_1(k)\langle \Delta \hat{E}_k \hat{A} \rangle, \tag{40}$$

where

$$\eta_0(k) = \frac{1}{2}\left\{\frac{1}{\exp(\beta(\langle \hat{E}_k \rangle + \varphi_k)) - 1} + \frac{1}{\exp(\beta(\langle \hat{E}_k \rangle - \varphi_k)) - 1}\right\}; \tag{41a}$$

$$\eta_1(k) = \frac{1}{2\varphi_k}\left\{\frac{1}{\exp(\beta(\langle \hat{E}_k \rangle + \varphi_k)) - 1} - \frac{1}{\exp(\beta(\langle \hat{E}_k \rangle - \varphi_k)) - 1}\right\}. \tag{41b}$$

From (40) it is possible to obtain the closed system of nonlinear integral equations. Putting $\hat{A} = 1$ in this equation, and by virtue of the symmetry involved, the mean value of the fluctuations vanishes: $\langle \Delta \hat{E}_k \rangle = 0$, so that we have the particle distribution

$$\langle \hat{n}_k \rangle = \eta_0(k). \tag{42}$$

We can then rewrite the long-range equation (40) in terms of the deviations of occupation-number operators, defined as

$$\Delta \hat{n}_k \equiv \hat{n}_k - \langle \hat{n}_k \rangle. \tag{43}$$

Thus,

$$\langle \Delta \hat{n}_k \hat{A} \rangle = \eta_1(k)\langle \Delta \hat{E}_k \hat{A} \rangle. \tag{44}$$

At this stage it is possible to obtain a closed system of nonlinear integral equations.

Putting in (44) $\hat{A} = \Delta \hat{n}_q$, we have the pair correlation function $\langle \Delta \hat{n}_k \Delta \hat{n}_q \rangle_c$, the index "*c*" denoting the true correlations ($k \neq q$) involved:

$$\begin{aligned} \langle \Delta \hat{n}_k \Delta \hat{n}_q \rangle_c &= \frac{\eta_1(k)}{\Omega} \sum_{\vec{p} \neq \vec{k}} V(\vec{k} - \vec{p})\langle \Delta \hat{n}_p \Delta \hat{n}_q \rangle \\ &= \eta_1(k) \frac{\langle (\Delta \hat{n}_q)^2 \rangle}{\Omega} V(\vec{k} - \vec{q}) + \frac{\eta_1(k)}{\Omega} \sum_{\vec{p} \neq \vec{k}} V(\vec{k} - \vec{p})\langle \Delta \hat{n}_p \Delta \hat{n}_q \rangle_c. \end{aligned} \tag{45}$$

For all $\vec{k}$-states $\eta_1(k)$ is negative; this is because there is a relative sign between the fluctuations in the number-of-particles operator and the fluctuations in the local-field operator. In the equilibrium state the particles prefer to energetically stay in the lowest-momentum state; so if the fluctuations in the local-field operator lead to an increase in the state energy, then the fluctuations in the number of particles lead to a decrease in the number of particles in this state. Accordingly, we must take the absolute value of $\eta_1(k)$ in our calculations; otherwise we shall have the wrong sign for the correlation functions.

From (45) we note that the correlation function depends on two terms. The first, the direct term, depends on the density fluctuations of the number of particles in the $\vec{q}$-state and the fluctuations in the energy state $|\vec{k}\rangle$ through the parameter $\eta_1(k)$. Through the direct term we see that the correlation function depends directly on the exchange potential, which in turn depends on three parameters: the modulus of $\vec{k}$ and of $\vec{q}$, and the angle between these two vectors. But the number of particles in state $|\vec{k}\rangle$ depends only on the modulus of $\vec{k}$, where all the directions are equally likely. Therefore, the fluctuations in the number of particles also depend on the modulus of $\vec{k}$. Then there is no physical reason for the corresponding correlation function to depend on the relative direction between $\vec{k}$ and $\vec{q}$. Accordingly, it is quite feasible to take the average exchange potential over all possible orientations:

$$\begin{aligned} \overline{V}(\vec{k}-\vec{q}) &\equiv \frac{1}{2}\int_{-1}^{1} V(\vec{k}-\vec{q})\, d\cos\theta \\ &= \frac{4\pi}{2}\int_0^\infty V(r)r^2\,dr \int_{-1}^{1} \frac{\sin(|\vec{k}-\vec{q}|r)}{|\vec{k}-\vec{q}|r}\, d\cos\theta \qquad (46a) \\ &= \frac{4\pi}{2}\int_0^\infty V(r)r^2\,dr \int_{-1}^{1} \frac{\sin((k^2+q^2-2kq\ \cos\theta)^{1/2}r)}{(k^2+q^2-2kq\ \cos\theta)^{1/2}r}\, d\cos\theta. \end{aligned}$$

(46b)

Integrating over the angle, we obtain the spherical Fourier-Bessel transform of the potential,

$$\overline{V}(\vec{k}-\vec{q}) \equiv V(k,q) = 4\pi\int_0^\infty V(r)r^2\,\frac{\sin(kr)}{kr}\,\frac{\sin(qr)}{qr}\,dr. \qquad (47)$$

The second term in the correlation function represents the exchange term and can be handled in a similar manner.

To determine the unknown value $\langle(\Delta\hat{n}_q)^2\rangle$ appearing in (45), we go back to the identity (11). We cannot use (44) because the operator $\hat{A}$ must commute with $\hat{b}_k^+$ and $\hat{b}_k$. Putting $\hat{A} = \hat{n}_q$ in that identity, we obtain

$$\langle \hat{b}_q^+(\beta)\hat{b}_q\hat{n}_q\rangle = \langle \hat{b}_q\hat{n}_q\hat{b}_q^+\rangle. \qquad (48)$$

With the commutation relation $[\hat{n}_q, \hat{b}_q^+] = \hat{b}_q^+$, this equation becomes

$$\langle \hat{b}_q^+(\beta)\hat{b}_q\hat{n}_q\rangle = \langle \hat{b}_q\hat{n}_q\hat{b}_q^+\rangle = \langle 1 + 2\hat{n}_q + \hat{n}_q^2\rangle. \qquad (49)$$

Since the local-field operator $\hat{E}_q$ commutes with the creation and annihilation operators, (48) assumes the form

$$\left\langle \hat{b}_q^+(\beta)\hat{b}_q\hat{n}_q \right\rangle = \left\langle \exp(\beta\hat{E}_q)\hat{b}_q^+\hat{b}_q\hat{n}_q \right\rangle. \tag{50}$$

From (49) and (50), we find $\langle \hat{n}_q^2 \rangle$:

$$\left\langle \hat{n}_q^2 \right\rangle = \left\langle \frac{1 + 2\hat{n}_q}{\exp(\beta\hat{E}_q) - 1} \right\rangle = \langle \hat{n}_q \rangle (1 + 2\langle \hat{n}_q \rangle) + 2\eta_1(q)\langle \Delta\hat{n}_q \Delta\hat{E}_q \rangle. \tag{51}$$

From this equation we can determine the quadratic fluctuations in the occupation numbers:

$$\langle (\Delta\hat{n}_q)^2 \rangle \equiv \left\langle \hat{n}_q^2 \right\rangle - \langle \hat{n}_q \rangle^2; \tag{52}$$

$$\langle (\Delta\hat{n}_q)^2 \rangle = \langle \hat{n}_q \rangle (1 + \langle \hat{n}_q \rangle) + \frac{2\eta_1(q)}{\Omega} \sum_{\vec{p}} V(\vec{p} - \vec{q}) \langle \Delta\hat{n}_p \Delta\hat{n}_q \rangle_c. \tag{53}$$

Putting $\hat{A} = \Delta\hat{E}_k$ in (44), we obtain

$$\eta_1(k)\varphi_k^2 = \frac{1}{\Omega} \sum_{\vec{p}} V(\vec{k} - \vec{p}) \langle \Delta\hat{n}_k \Delta\hat{n}_p \rangle_c. \tag{54}$$

We now have the closed system of nonlinear integral equations consisting of $\langle \hat{E}_k \rangle$, $\langle \hat{n}_k \rangle$, $\langle (\Delta\hat{n}_k)^2 \rangle$, $\langle \Delta\hat{n}_k \Delta\hat{n}_q \rangle_c$, φ_k. These nonlinear equations will be solved numerically by Gaussian quadrature (Ali, 1997; Bishop *et al.*, 1977; Burden and Faires, 1993; Ghassib *et al.*, 1976).

To calculate the thermodynamic properties of the system, the grand partition function Q should be derived. We start from the usual expression

$$\begin{aligned} Q = Tr(\exp(-\beta\hat{H})) &= \sum_{n_p} \exp\left(-\beta \sum_{\vec{p}} \hat{E}_p \hat{n}_p \right) \\ &= \prod_{\vec{p}} \sum_{n_p} \exp(-\beta\hat{E}_p\hat{n}_p). \end{aligned} \tag{55}$$

It is simplest to take the logarithms of both sides:

$$\begin{aligned} \ln Q = \ln(Tr[\exp(-\beta\hat{H})]) &= \ln\left[\sum_{n_p} \exp\left(-\beta \sum_{\vec{p}} \hat{E}_p \hat{n}_p \right) \right] \\ &= \ln\left[\prod_{\vec{p}} \sum_{n_p} \exp(-\beta\hat{E}_p\hat{n}_p) \right] \\ &= -\sum_{\vec{p}} \ln[1 - \exp(-\beta\hat{E}_p)]. \end{aligned} \tag{56}$$

Using identity (38), we have

$$\ln Q = -\sum_{\vec{p}} [q_0(p) + q_1(p)\Delta\hat{E}_p]. \tag{57}$$

Taking into account the symmetry of the two eigenvalues of the operator $\Delta\hat{E}_p$, we finally get from (57)

$$\ln Q = -\sum_{\vec{p}} q_0(p); \tag{58}$$

$$q_0(p) = \frac{1}{2}\ln[(1 - \exp(-\beta\langle\hat{E}_p\rangle - \beta\varphi_p))(1 - \exp(-\beta\langle\hat{E}_p\rangle + \beta\varphi_p))]. \tag{59}$$

The mean internal energy $\langle\hat{H}\rangle$ is, then,

$$\langle\hat{H}\rangle = -\frac{\partial \ln Q}{\partial\beta},$$

or

$$U = \langle\hat{H}\rangle = \sum_{\vec{p}} \langle\hat{n}_p\hat{E}_p\rangle = \sum_{\vec{p}} \left(\langle\hat{E}_p\rangle\langle\hat{n}_p\rangle + \varphi_p^2\eta_1(p)\right). \tag{60}$$

From the grand partition function and the grand internal energy it is straightforward to evaluate all other thermodynamic properties.

4. CALCULATIONS

The above set of nonlinear integral equations has been solved numerically by iteration for a realistic interhelium potential. Throughout our calculations a natural system of units has been used, such that $\hbar = 1 = m$, the conversion factor being $\frac{\hbar^2}{m} = 12.120048$ K Å^2.

For convenience, the central He–He potential can be expressed in the form

$$V(r) = Af(r), \tag{61}$$

where A is a factor determining the strength (amplitude) of the potential and *f(r)* is a “shape function” describing the overall shape of the potential (Ghassib, 1984). For liquid ^{4}He, *V(r)* could be one of several highly-acclaimed realistic potentials: FDD-1, MDD-2, S, B (Bishop *et al.*, 1977) and HFDHE2 (Ali, 1997; Aziz *et al.*, 1979). HFDHE2 seems to be the most reliable. Here, however, we have chosen the FDD-1 potential of Frost and Musulin (Bruch and McGee, 1967, 1970) as our archetype potential because it retains most of the desirable features of HFDHE2 but is simpler to handle.

Although these potentials are well known, we now give the explicit forms of FDD-1 and HFDHE2 potentials for completeness and for reference purposes:
(a) FDD-1:

$$V(r) = \begin{cases} -\varepsilon\left[1 + c\left(1 - \dfrac{r_m}{r}\right)\right]\exp\left[c\left(1 - \dfrac{r}{r_m}\right)\right], & r \le r_S \\ -\left(\dfrac{C_6}{r^6} + \dfrac{C_8}{r^8}\right), & r \ge r_S \end{cases}$$

$\varepsilon = 12.54$ K; $c = 8.01$; $r_s = 3.5$ Å; $r_m = 2.98$ Å; $C_6 = 10213.8$ K Å^{-6}; $C_8 = 27671.4$ K Å^{-8}.
(b) HFDHE2:

$$V(r) = \varepsilon V^*(x),$$

$$V^*(x) = A\exp(-\alpha x) - \left\{\frac{C_6}{x^6} + \frac{C_8}{x^8} + \frac{C_{10}}{x^{10}}\right\}F(x);$$

$$F(x) = \begin{cases} \exp -\left(\left[\dfrac{D}{x} - 1\right]^2\right), & x < D \\ 1, & x \ge D \end{cases}$$

$x \equiv \frac{r}{r_m}$; $D = 1.24$; $A = 0.554 \times 10^6$; $\alpha = 13.35$; $C_6 = 1.373$; $C_8 = 0.425$; $C_{10} = 0.178$; $\varepsilon = 10.8$ K.

The chemical potential of the system also plays a crucial role in the formalism. From statistical mechanics the number of particles of each single-particle state $|\vec{k}\rangle$ for the noninteracting system is given by

$$\langle \hat{n}_k \rangle = \frac{1}{\exp(\beta\varepsilon(k)) - 1}, \tag{62}$$

where $\varepsilon(k)$ is the kinetic energy of the system incorporating the chemical potential. This expression is meaningful only if

$$\varepsilon(k) \equiv \frac{\hbar k^2}{2m} - \mu > 0; \tag{63}$$

otherwise the mean occupation number of particles will be negative. To satisfy (63) at all, the chemical potential must be less than the minimum value of the kinetic energy. But the kinetic energy might vanish; so that the chemical potential must always vanish or be negative.

According to our formalism, the number of particles in each single-particle state for interacting systems is defined as

$$\langle \hat{n}_k \rangle = \frac{1}{2}\left\{\frac{1}{\exp(\beta(\langle \hat{E}_k \rangle + \varphi_k)) - 1} + \frac{1}{\exp(\beta(\langle \hat{E}_k \rangle - \varphi_k)) - 1}\right\}. \tag{64}$$

This equation is meaningful only if

$$\langle \hat{E}_k \rangle \pm \varphi_k \geq 0. \tag{65}$$

Based on this condition, the chemical potential is

$$\mu = -\min\left(\frac{\hbar^2 k^2}{2m} + \frac{1}{\Omega} \sum_{\vec{p} \neq \vec{k}} V(\vec{p} - \vec{k}) \langle \hat{n}_p \rangle \pm \varphi_k \right). \tag{66}$$

The minus sign arises here from physical considerations; the chemical potential for neutral many-bosonic systems is negative. At all $|\vec{k}\rangle$, φ_k is positive. Then (66) becomes

$$\mu = -\min\left(\frac{\hbar^2 k^2}{2m} + \frac{1}{\Omega} \sum_{\vec{p} \neq \vec{k}} V(\vec{p} - \vec{k}) \langle \hat{n}_p \rangle - \varphi_k \right). \tag{67}$$

This result is valid only at low temperatures, $T \leq T_\lambda$, where if we add a boson to the system it can go to the lowest-momentum state.

5. RESULTS AND DISCUSSION

5.1. The Weakly Interacting System: Liquid ^{4}He in Vycor Glass

For a fixed $f(r)$ in (61), it turns out that our closed system of integral equations converges only when $A \leq 1 \times 10^{-4}$. This means that the present approximation is valid for dilute systems only, such as liquid ^{4}He in Vycor glass.

With $A = 10^{-4}$, then, the following quantities have been computed:

(i) the relative fluctuation of state $|\vec{k}\rangle$; that is, $\frac{\varphi_k}{\langle \hat{E}_k \rangle}$,
(ii) the fluctuations of the local-field operator,
(iii) $\frac{\langle n_k \rangle}{\langle n_1 \rangle}$.

To examine the dependence of these quantities on temperature, they have been computed for two temperatures: $T_1 = 0.5$ K and $T_2 = 2$ K.

Figure 1 shows the relative fluctuations at temperatures T_1 and T_2. The basic features are clear: these have maximum values (≈ 0.55 at T_1 and ≈ 0.60 at T_2) at the lowest-momentum state and vanish asymptotically at higher states. Further, they decrease rapidly at T_1, whereas at T_2 they extend to higher states. Thus, the higher orders of these fluctuations become more important as the temperature increases.

Figures 2 and 3 show the fluctuations of the local-field operator at temperatures T_1 and T_2, respectively. Again, the corresponding maximum values occur at the lowest-momentum state ($\approx 0.35 \times 10^{-2}$ Å^{-2} at T_1 and 0.24×10^{-1} Å^{-2} at T_2) and vanish at higher states.

How can we explain these results? From (54) the fluctuations in state $|\vec{k}\rangle$ depend on the spherical Fourier-Bessel transform of the potential, which decreases

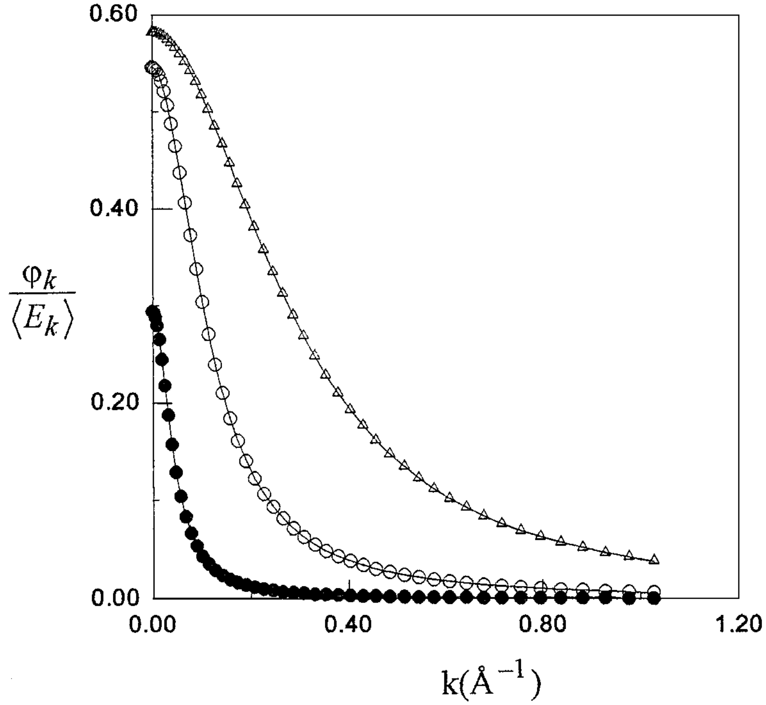

Fig. 1. The relative fluctuations ($\varphi k/\langle E_k \rangle$) for the FDD-1 potential with strength factor $A = 10^{-4}$ at temperatures $T_1 = 0.5$ K (o), $T_2 = 2$ K (Δ), and with strength factor $A = 10^{-6}$ at temperature $T_2 = 2$ K ($\bullet$), versus the relative momentum (k).

when k increases. This can be seen even more clearly from (47). Therefore, the fluctuations decrease as k increases.

Further, Figs. 1–3 show that, as the temperature increases, the relative fluctuations and the fluctuations increase and extend to higher states. This is not surprising since, as the temperature increases, more and more bosons are depleted from the lowest-momentum state to higher states.

This can be deduced at once from Fig. 4: the width of the distribution of the number of bosons in the low-lying excited states relative to the first excited state increases with temperature. Now, the interaction term of the local-field operator in state $|\vec{k}\rangle$ depends on the interaction between this state and the other occupied states. As the temperature increases, the number of occupied states increases; so that the interaction term in the local-field operator also increases. As a result, the fluctuations of the local-field operator will increase. Alternatively, from (54), the fluctuations in the local-field operator in state $|\vec{k}\rangle$ depend on the correlation function between the fluctuations in the number of bosons in this state and those

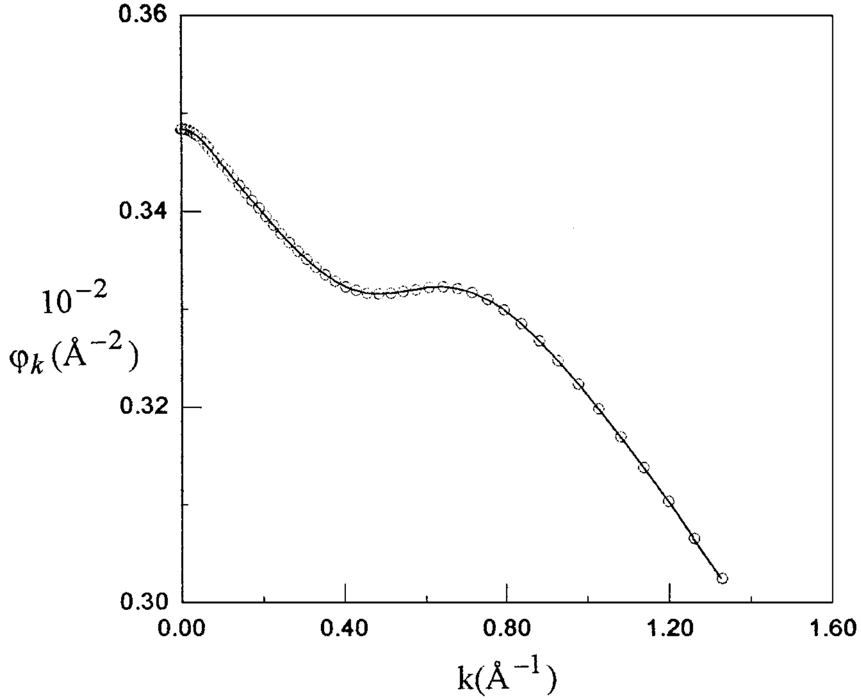

Fig. 2. The fluctuations (φ_k) for the FDD-1 potential with strength factor $A = 10^{-4}$ at temperature $T_2 = 0.5$ K versus the relative momentum (k).

corresponding to the other occupied states. As the temperature increases, more states become occupied; so that the fluctuations in the local-field operator of state $|\vec{k}\rangle$ increase.

In passing, it is noted that the fluctuations at low T (Fig. 2) are characterized by a *dip* around 0.4 Å^{-1}. This may be attributed to "two-fluidity" or two different excitations in the system. At high $T(\geq 2$ K) the *dip* disappears (Fig. 3).

Finally, to explore the effect of the strength factor on the relative fluctuations and the distribution of particles, these quantities have been computed when $A = 10^{-6}$ and $T = T_2$. The results are shown in Figs. 1 and 4. It is clear from Fig. 1 that the relative fluctuations have a maximum value (≈ 0.3) at the lowest-momentum state, which is less than the corresponding value in the previous case. They also decrease faster than in that case. The physical implication is that the strength of the potential plays a crucial role here, as expected. Likewise, Fig. 4 shows that the width of the distribution of the depleted bosons in the low-lying excited states shrinks as compared to the previous cases—again as expected.

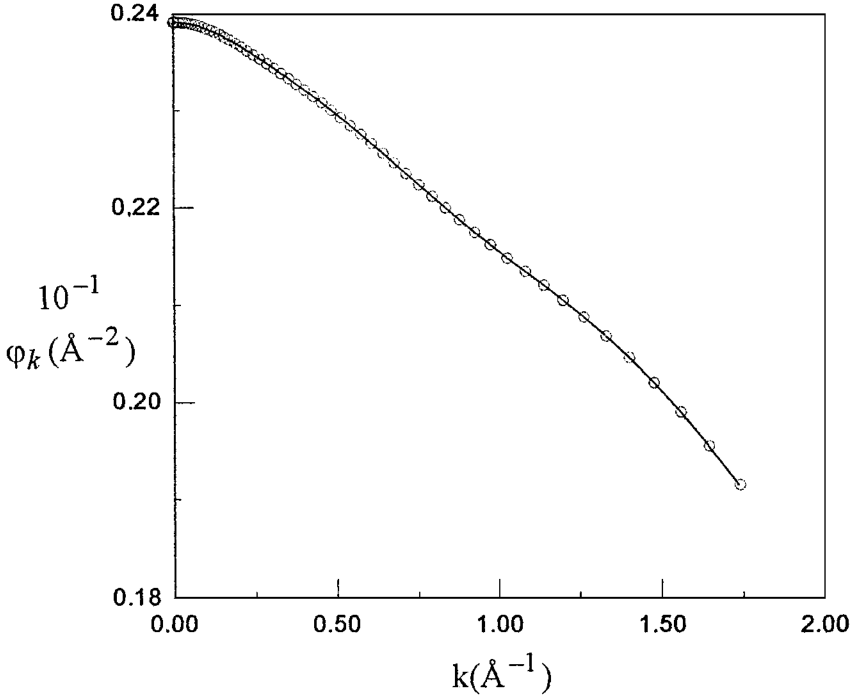

Fig. 3. The fluctuations (φ_k) for the FDD-1 potential with strength factor $A = 10^{-4}$ at temperature $T_2 = 2$ K versus the relative momentum (k).

A major aim of any statistical–mechanical or many-body theory is to derive the thermodynamic properties of the system under examination. This aim is achieved here within the SFA for the FDD-1 potential with strength factor $A = 10^{-4}$. The underlying physical system is the dilute weakly interacting "liquid" ^{4}He, which can be realized in practice in the form of liquid ^{4}He in Vycor glass (Crooker, 1984; Reppy *et al.*, 1984). The following results are therefore offered here for experimentalists as definite predictions for the thermodynamic properties of this system.

The starting point is the grand partition function Q derived in (58). From (60) the grand internal energy (the internal energy incorporating the chemical potential) per unit volume can be computed numerically for the preceding system. The corresponding plot is shown in Fig. 5 as a function of temperature.

The pressure is given by (Huang, 1987; Pathria, 1992)

$$P = k_B T\,\frac{\ln Q}{\Omega}. \tag{68}$$

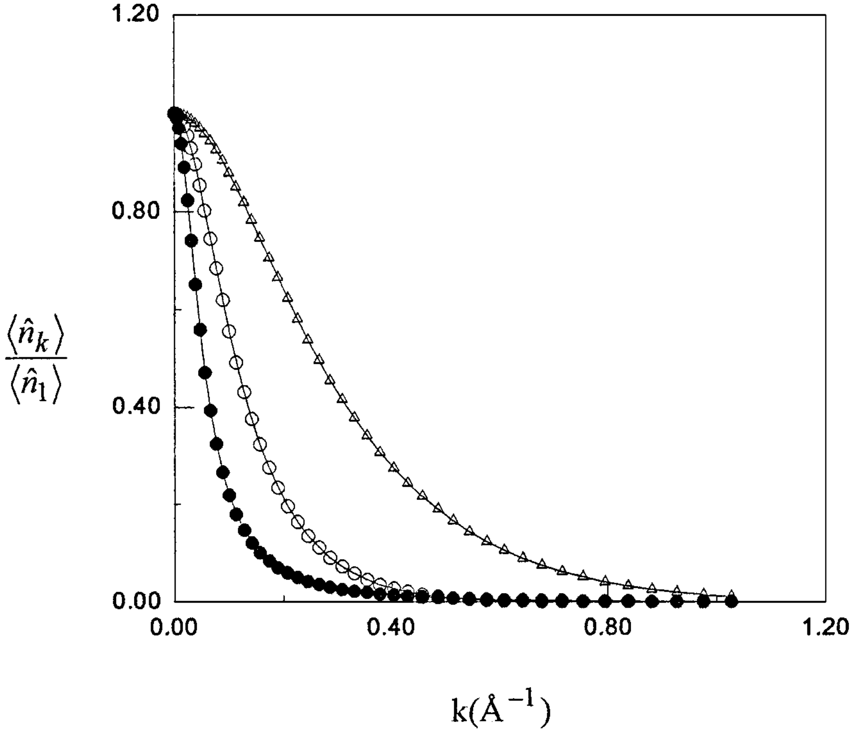

Fig. 4. Distribution of the number of particles in the depletion states relative to the first depletion state ($\langle \hat{n}_k \rangle / \langle \hat{n}_1 \rangle$), for the FDD-1 potential with strength factor $A = 10^{-4}$ at temperatures $T_1 = 0.5$ K (o), $T_2 = 2$ K (Δ), and with strength factor $A = 10^{-6}$ at temperature $T_2 = 2$ K (•), versus the relative momentum (k).

This is shown in Fig. 6 as a function of temperature; it represents the equation of state of the system.

Other thermodynamic properties can readily be obtained in terms of the pressure and the grand internal energy. From the first law of thermodynamics we have:

$$U = -P\Omega + TS, \tag{69}$$

where U here represents the grand internal energy—consistent with (58). From (68) and (69) we have for the entropy of the system

$$S = \frac{U}{T} + k_{\mathrm{B}} \ln Q. \tag{70}$$

The entropy per unit volume is plotted in Fig. 7 as a function of temperature.

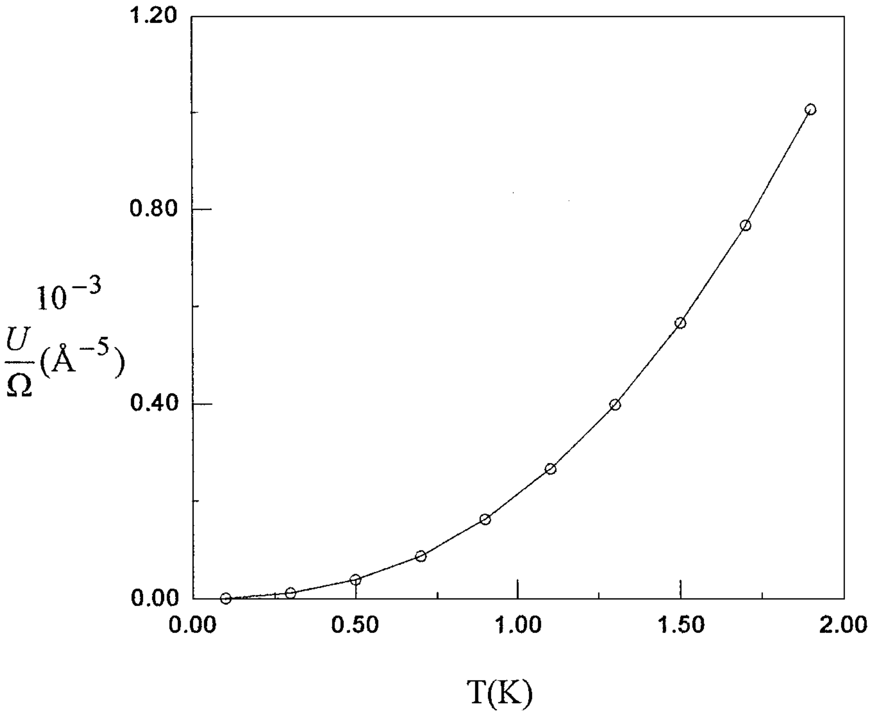

Fig. 5. The grand mean energy per unit volume (U/Ω) for liquid ^{4}He in Vycor glass as a function of temperature (T).

Finally, the specific heat capacity of the system at constant volume is given by the usual definition:

$$C_v = T\left(\frac{\partial S}{\partial T}\right)_\Omega = \left(\frac{\partial U}{\partial T}\right)_\Omega . \tag{71}$$

This quantity per unit volume is presented in Fig. 8 as a function of temperature.

The functional dependence on T of the above thermodynamic quantities can be obtained by straightforward fitting procedures. The results are

1. the grand internal energy $U \propto T^{2.45}$,
2. the pressure $P \propto T^{2.41}$,
3. the entropy $S \propto T^{1.47}$,
4. the specific heat capacity $C_v \propto T^{1.43}$.

For the ideal system (Pathria, 1992) both U and $P \propto T^{2.5}$, whereas S and $C_v \propto T^{1.5}$. Thus, the thermodynamic properties for liquid ^{4}He in Vycor glass have almost the same temperature dependence as for the ideal system. The small

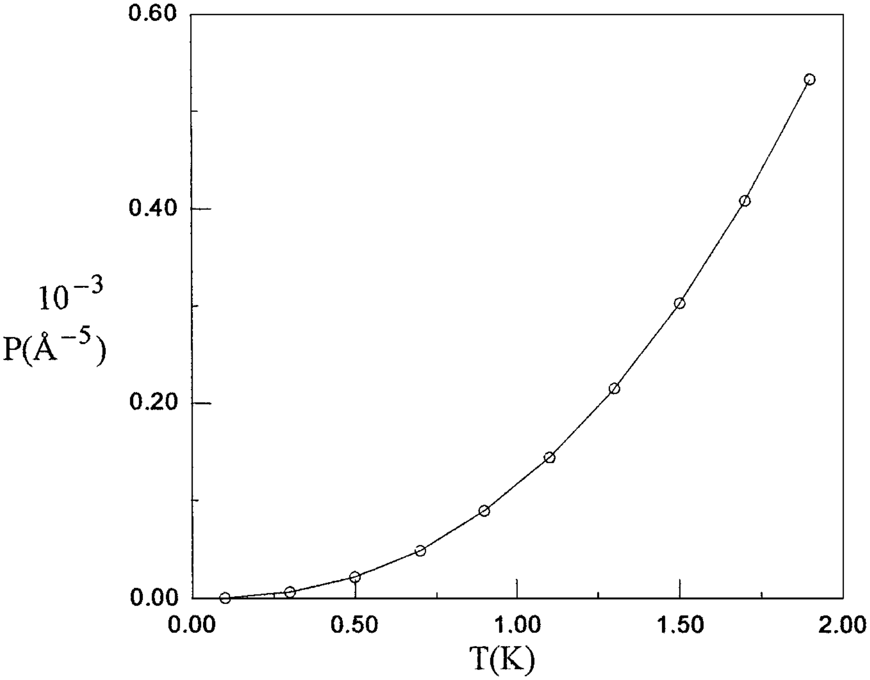

Fig. 6. The pressure (P) for liquid ^{4}He in Vycor glass as a function of temperature (T).

deviation of the T-exponent arises from the weak interaction of the ^{4}He system in Vycor glass, as expected.

5.2. The Strongly Interacting System: Liquid ^{4}He

In the preceding calculations the fluctuations of the local-field operator have been found to increase as the strength of the potential increases. Thus, for strongly interacting systems, the fluctuations cannot be cosidered small compared to the mean value of the local-field operator. We have therefore to start again from (36) to derive the closed system of nonlinear integral equations valid for such systems.

Equation (36) is first rewritten:

$$\langle \hat{n}_k(\exp \beta(\langle \hat{E}_k \rangle + \Delta \hat{E}_k) - 1)\hat{A} \rangle = \langle \hat{A} \rangle. \tag{72}$$

Next, (38) is used to write $\exp(\beta \Delta \hat{E}_k)$ as linear in terms of the fluctuations of the local-field operator. Equation (72) then becomes

$$\langle \hat{n}_k(\zeta_0(k) + \zeta_1(k)\Delta \hat{E}_k - 1)\hat{A} \rangle = \langle \hat{A} \rangle, \tag{73}$$

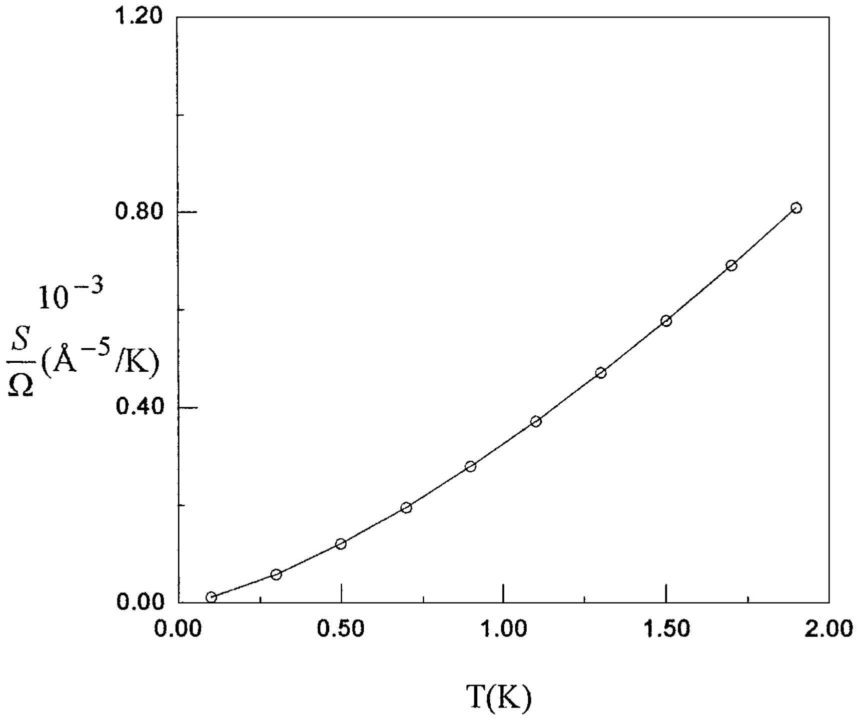

Fig. 7. The entropy per unit volume (S/Ω) for liquid ^{4}He in Vycor glass as a function of temperature (T).

where

$$\zeta_0(k) \equiv \exp(\beta\langle\hat{E}_k\rangle)\cosh(\beta\varphi_k)\,; \tag{74a}$$

$$\zeta_1(k) \equiv \exp(\beta\langle\hat{E}_k\rangle)\frac{\sinh(\beta\varphi_k)}{\varphi_k}. \tag{74b}$$

In (73) it is more convenient to express the operator $\hat{n}_k$ in terms of its mean value and its fluctuations:

$$\hat{n}_k = \langle\hat{n}_k\rangle - \Delta\hat{n}_k. \tag{75}$$

As already stressed, if the local-field operator is written as its mean value *plus* the fluctuation as has indeed been done, the number-of-particles operator must be written as its mean value *minus* the fluctuation. The underling physical meaning is that, if the fluctuation in the local-field operator leads to an increase in the energy of

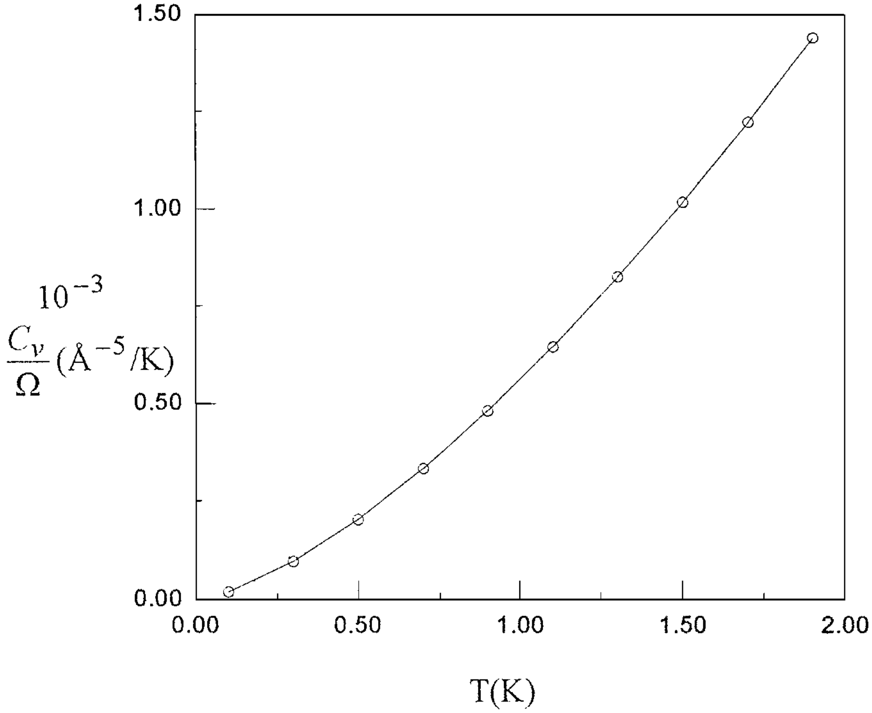

Fig. 8. The specific heat capacity per unit volume (C_V/Ω) for liquid ^{4}He in Vycor glass as a function of temperature (T).

the state $|\vec{k}\rangle$, then the number of particles in this state must decrease. Accordingly, (73) becomes

$$\langle \Delta\hat{n}_k \hat{A}\rangle = \left(\langle \hat{n}_k\rangle - \frac{1}{\zeta_0(k)-1}\right)\langle \hat{A}\rangle + \frac{\zeta_1(k)\langle \hat{n}_k\rangle}{\zeta_0(k)-1}\langle \Delta\hat{E}_k\hat{A}\rangle$$
$$- \frac{\zeta_1(k)}{\zeta_0(k)-1}\langle \Delta\hat{n}_k\Delta\hat{E}_k\hat{A}\rangle. \tag{76}$$

Relative to (40), (76) has two additional terms: the first term, which depends on the mean value of the operator $\hat{A}$; and the last, which depends on the triplet correlations relating the fluctuation in the number of particles to that in the local-field operator and to the operator $\hat{A}$. From (76) putting $\hat{A} = 1$, we have

$$\langle \hat{n}_k\rangle = \frac{1 + \zeta_1(k)\langle \Delta\hat{n}_k\Delta\hat{E}_k\rangle}{\zeta_0(k)-1}. \tag{77}$$

Again, putting $\hat{A} = \Delta\hat{E}_k$, we obtain

$$\langle \Delta\hat{n}_k \Delta\hat{E}_k \rangle = \frac{\zeta_1(k)\langle \hat{n}_k \rangle}{\zeta_0(k) - 1}\varphi_k^2. \tag{78}$$

From (77) and (78),

$$\langle \hat{n}_k \rangle = \frac{\zeta_0(k) - 1}{(\zeta_0(k) - 1)^2 - \zeta_1^2(k)\varphi_k^2}. \tag{79}$$

To close our system of nonlinear integral equations, $\langle \Delta\hat{n}_k \Delta\hat{E}_k \rangle$ is determined from (35):

$$\langle \Delta\hat{n}_k \Delta\hat{E}_k \rangle = \frac{1}{\Omega}\sum_{\vec{p}} V(\vec{p} - \vec{k})\langle \Delta\hat{n}_k \Delta\hat{n}_p \rangle_c. \tag{80}$$

The pair correlation function $\langle \Delta\hat{n}_k \Delta n_q \rangle_c$ is found by putting in (76) $\hat{A} = \Delta\hat{n}_q$, where $q \neq k$:

$$\langle \Delta\hat{n}_k \Delta\hat{n}_q \rangle_c = \frac{\zeta_1(k)\langle \hat{n}_k \rangle}{\zeta_0(k) - 1}\langle \Delta\hat{n}_q \Delta\hat{E}_k \rangle - \frac{\zeta_1(k)}{\zeta_0(k) - 1}\langle \Delta\hat{n}_k \Delta\hat{E}_k \Delta\hat{n}_q \rangle. \tag{81}$$

The last term on the right hand side of this equation represents the three-body correlation function whose contribution is assumed to be small compared to that of the pair correlation function. Therefore, the last term in (81) can be neglected. It follows that

$$\langle \Delta\hat{n}_k \Delta\hat{n}_q \rangle_c \approx \frac{\zeta_1(k)\langle \hat{n}_k \rangle}{\zeta_0(k) - 1}\frac{1}{\Omega}\sum_{\vec{p}\neq\vec{k}} V(\vec{p} - \vec{k})\begin{pmatrix} \langle \Delta\hat{n}_q \Delta\hat{n}_p \rangle_c \\ + \delta_{qp}\langle (\Delta\hat{n}_p)^2 \rangle \end{pmatrix} \tag{82a}$$

$$\equiv \frac{\zeta_1(k)\langle \hat{n}_k \rangle}{\zeta_0(k) - 1} f(k, q)_c. \tag{82b}$$

To calculate $\langle (\Delta\hat{n}_q)^2 \rangle$ we must go back to (11); we cannot use (76) since $\hat{A}$ in this equation must commute with both creation and annihilation operators. From (11) we find

$$\left\langle 1 + 2\hat{n}_q + \hat{n}_q^2 \right\rangle = \left\langle \hat{n}_q^2(\zeta_0(q) + \zeta_1(q)\Delta\hat{E}_q) \right\rangle. \tag{83}$$

Substituting $\hat{n}_q^2 = \langle \hat{n}_q \rangle^2 + (\Delta\hat{n}_q)^2 - 2\langle \hat{n}_q \rangle \Delta\hat{n}_q$ in (83) and performing simple mathematical manipulations, we have

$$\langle (\Delta\hat{n}_q)^2 \rangle = \langle \hat{n}_q \rangle^2 + \frac{1}{\zeta_0(q) - 1}. \tag{84}$$

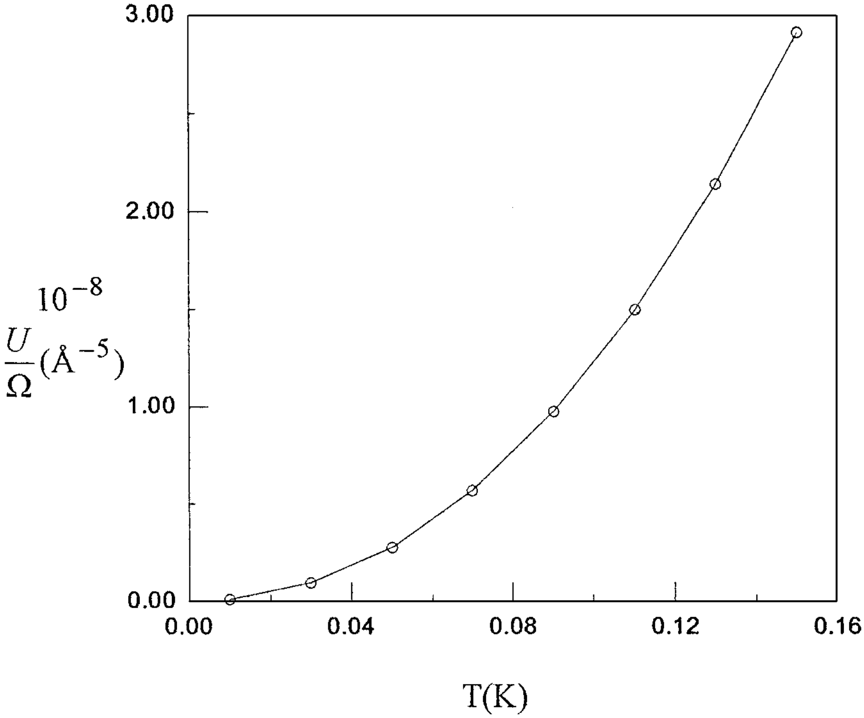

Fig. 9. The grand mean energy per unit volume (U/Ω) for liquid ^{4}He as a function of temperature (T).

From (78) and (80) we can calculate the fluctuation in the local-field operator:

$$\frac{\zeta_1(k)\langle\hat{n}_k\rangle}{\zeta_0(k)-1}\varphi_k^2 = \frac{1}{\Omega}\sum_{\vec{p}} V(\vec{p}-\vec{k})\langle\Delta\hat{n}_k\Delta\hat{n}_p\rangle_c. \tag{85a}$$

To avoid the singularity at higher states, it is more convenient to rewrite (85a) in the form

$$\varphi_k^2 = \frac{1}{\Omega}\sum_{\vec{p}} V(\vec{p}-\vec{k})\, f(k,p)_c. \tag{85b}$$

Finally, the chemical potential should be determined. Physically, the number of particles in each single-particle state (77) should be positive. This condition is satisfied if

$$\zeta_0(k) \geq 1. \tag{86}$$

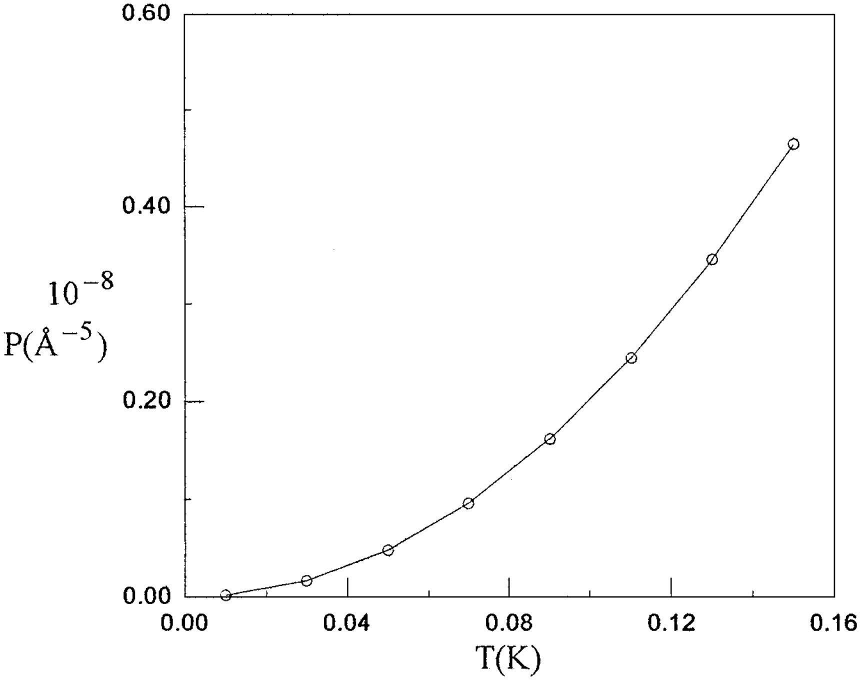

Fig. 10. The pressure (P) for liquid ^{4}He as a function of temperature (T).

In addition, since μ for bosonic systems should be negative, we have

$$\mu = -\min\left[\frac{\hbar^2k^2}{2m} + \frac{1}{\Omega}\sum_{\vec{p}\neq\vec{k}} V(\vec{p}-\vec{k})\,\langle\hat{n}_p\rangle + k_{\mathrm{B}}T\,\ln\{\cosh(\beta\varphi_k)\}\right]. \tag{87}$$

Our new set of nonlinear integral equations can be solved numerically. To compute the thermodynamic properties, the grand partition function should be derived. This can be achieved by repeating the same mathematical manipulations of (55) up to (58). To complete the thermodynamic description of the system the grand mean energy of the system is needed. This is given by

$$U = \langle\hat{H}\rangle = \sum_{\vec{k}}\langle\hat{n}_k\hat{E}_k\rangle = \sum_{\vec{k}}(\langle\hat{n}_k\rangle\langle\hat{E}_k\rangle - \langle\Delta\hat{n}_k\Delta\hat{E}_k\rangle), \tag{88}$$

or

$$U = \sum_{\vec{k}}\left(\langle\hat{n}_k\rangle\langle\hat{E}_k\rangle - \frac{\zeta_1(k)\langle\hat{n}_k\rangle}{\zeta_0(k)-1}\varphi_k^2\right). \tag{89}$$

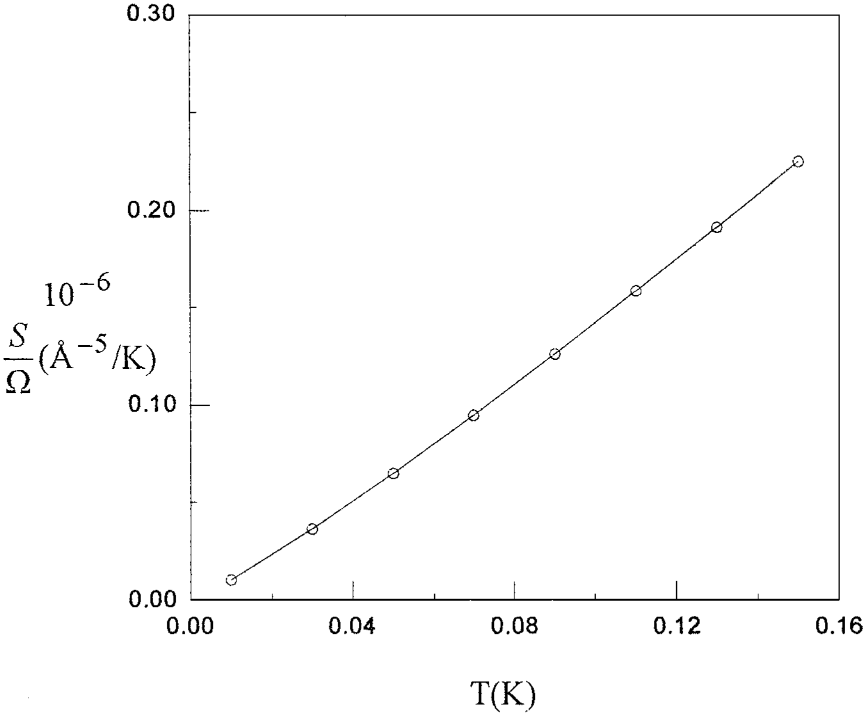

Fig. 11. The entropy per unit volume (S/Ω) for liquid ^{4}He as a function of temperature (T).

Figures 9–12 show the plots of the grand mean energy per unit volume, the pressure, the entropy per unit volume, and the specific heat capacity per unit volume, respectively, as functions of temperature up to 0.15 K. At higher temperatures the closed system of nonlinear integral equations diverges by virtue of the increasing fluctuations of the local-field operator.

By fitting the above thermodynamic data, we find that

1. the grand internal energy $U \propto T^{2.15}$,
2. the pressure $P \propto T^{2.1}$,
3. the entropy $S \propto T^{1.21}$,
4. the specific heat capacity $C_v \propto T^{1.14}$.

Comparing these to the corresponding results obtained for liquid ^{4}He in Vycor glass, we conclude that the temperature dependence of the above thermodynamic properties decreases when the strength of the interaction potential increases.

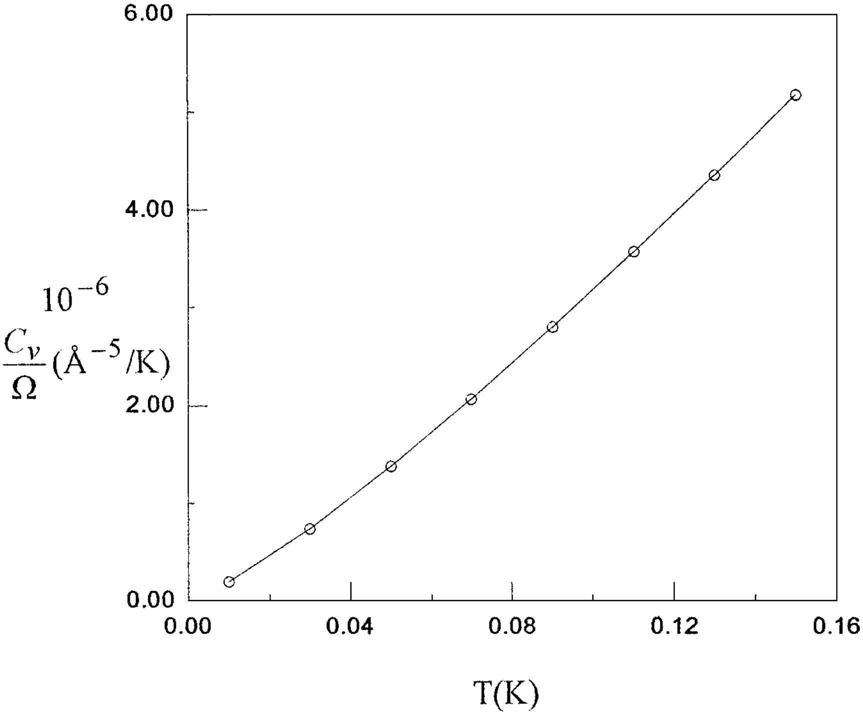

Fig. 12. The specific heat capacity per unit volume (C_V/Ω) for liquid ^{4}He as a function of temperature (T).

In passing, it is gratifying to note that the above results for S and C_V, when extrapolated to $T = 0.2$ K, agree very well with the available experimental results (Bendt *et al.*, 1959).

6. CONCLUSIONS

Neutral many-bosonic systems have been studied for the first time within the framework of the SFA, which hinges on the replacement of the *square* of the local-field operator with its mean value. To satisfy the basic condition that the local-field operator must commute with both the annihilation and creation operators, it turns out that the interparticle interaction in momentum space must vanish when the relative momentum vanishes. This, in turn, has necessitated rewriting the Hamiltonian so as to conform with this condition.

An algorithmic version has been developed, and extensive calculations have been undertaken for the relative fluctuations, fluctuations, distribution of particles, and thermodynamic properties of the dilute, weakly interacting liquid ^{4}He in Vycor glass. The results should present a challenge to experimentalists for vindication and verification. In addition, the thermodynamic properties of the strongly interacting liquid ^{4}He have been calculated, albeit for low temperatures only ($\leq$0.15 K).

The basic achievements of this paper are as follows:

1. The full derivation of the SFA for the first time for neutral many-bosonic systems.
2. The development of the corresponding algorithmic version for solving the closed system of nonlinear integral equations involved.
3. The prediction of the thermodynamic properties of the weakly interacting liquid ^{4}He in Vycor glass.
4. The calculation of the thermodynamic properties of the strongly interacting liquid ^{4}He at low temperatures ($\leq$0.15 K).

Several possible extensions and generalizations are implied by this work.

The first is the reformulation of the present framework for higher orders of fluctuations. The resulting theory should be more suitable for strongly interacting systems at arbitrary temperatures. The second is to take into account the triplet correlation function in deriving the closed system of nonlinear integral equations. The third is to study the thermohydrodynamic (or hydrothermodynamic) properties of neutral many-bosonic systems based on the present formulation of the SFA. The fourth is the application of the SFA to neutral weakly and strongly interacting many-fermionic systems. The charged Fermi system (the electronic fluid) has already been studied within the SFA scheme (Mash'al, 2000). The fifth problem is the reformulation of the present framework for confined bosonic as well as fermionic systems.

In all these problems helpful hints and guidelines could be furnished by previous published attempts within other formulations.

ACKNOWLEDGMENTS

Two of the authors (H.B.G. and R.R.N.) gratefully acknowledge partial financial support from the Deanship of Scientific Research, University of Jordan, under Grant No. 5/3/2/4635 dated 5/12/1998, and renewed on 1/3/2000.

REFERENCES

Ali, S. N. (1997). The helium-helium interaction in vacuum: A reconsideration. M.Sc. Thesis, University of Jordan, Amman, Jordan.

Aziz, R. A., Nain, V. P. S., Carley, J. S., Taylor, W. L., and McConville, G. T. (1979). *Journal of Chemical Physics* **70**, 4330.

Bendt, P. J., Cowan, R. D., and Yarnell, J. L. (1959). *Physical Review* **113**, 1386.

Bishop, R. F., Ghassib, H. B., and Strayer, M. R. (1977). *Journal of Low Temperature Physics* **26**, 669.

Bruch, L. and McGee, I. (1967). *Journal of Chemical Physics* **46**, 2959.

Bruch, L. and McGee, I. (1970). *Journal of Chemical Physics* **52**, 5884.

Burden, R. L. and Faires, J. D. (1993). *Numerical Analysis*, 5th edn., PWS Publishing Company, Boston.

Crooker, B. C. (1984). Studies of the flow properties of superfluid ^{3}He and of the superfluid density of ^{4}He films in restricted geometries PhD. Thesis, Cornell University, Ithaca, New York.

Feynman, R. P. (1972). *Statistical Mechanics*, Benjamin, Reading, MA.

Galasiewicz, Z. M. (1971). *Helium-4*, Pergamon Press, Oxford.

Ghassib, H. B., Bishop, R. F., and Strayer, M. R. (1976). *Journal of Low Temperature Physics* **23**, 393.

Ghassib, H. B. (1984). *Zeitschrift für Physik B-Condensed Matter* **56**, 91.

Ghassib, H. B. and Khudeir, A. M. (1986). *International Journal of Theoretical Physics* **25**, 255.

Girish, S. S. and Yia-Chung, C. (1998). *Physical Review B* **57**, 15144.

Huang, K. (1987). *Statistical Mechanics*, 2nd edn., Wiley, New York.

Mash'al, L. A. (2000). Electronic correlations: New model calculations. M.Sc. Thesis, University of Jordan, Amman, Jordan.

Nigmatullin, R. R. and Toboev, V. A. (1989). *Theoretical and Mathematical Physics* **80**, 94.

Nigmatullin, R. R., Khamzin, A. A., and Ghassib, H. B. (2000a). *Solid State Communications* **113**, 257.

Nigmatullin, R. R., Khamzin, A. A., and Ghassib, H. B. (2000b). *International Journal of Theoretical Physics* **39**, 405.

Nigmatullin, R. R., Khamzin, A. A., and Ghassib, H. B. (2000c). *Physical Review E* **61**, 3441.

Pathria, R. K. (1992). *Statistical Mechanics*, 3rd edn., Pergamon Press, Oxford.

Reppy, J. D. (1984). *Physica* **126B**, 335.

Tilley, D. R. and Tilley, J. (1974). *Superfluidity and Superconductivity*, 2nd edn., Van Nostrand-Reinhold, New York.

Wilks, J. (1967). *The Properties of Liquid and Solid Helium.* Clarendon Press, Oxford.